

# Domain Wall Filters

Oliver Baer (HU), Rajamani Narayanan (FIU),  
Herbert Neuberger (Rutgers), Oliver Witzel (HU)

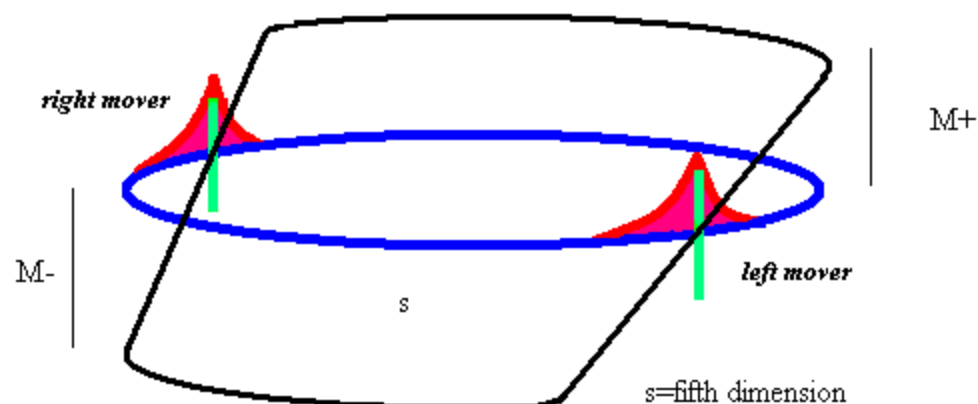
# Review I

Let us start around year 4  
BBS (before Blum+Soni).

We have Kaplan's original  
paper:

- 1) There is a  $A_5(x_\mu, s)$ .
- 2)  $A_\mu(x_\nu, s)$  depends on  $s$

FIG. 1



## Review II

There also is a paper by Slavnov+Frolov, using 4D gauge theory, but with what amounts to an infinite flavor space.

NN: Kaplan and Slavnov+Frolov are the same if:

0) We send one wall to infinity, keeping only the vicinity of the other wall finite.

1) and:  $A_5 = 0, \frac{\partial A_\mu}{\partial s} = 0$

## Review III

- 2) The 4D action, with  $s$  viewed as a continuous or discrete flavor space has the structure:

$$\gamma_\mu D_\mu + M$$

where  $M$  has an index

$$MM^\dagger > 0, \dots \dim(Ker(M)) = 1$$

## Review IV

- A) This explains stability against radiative corrections on the lattice.
- B) This prevents the extra gauge degrees of freedom from creating extra walls.

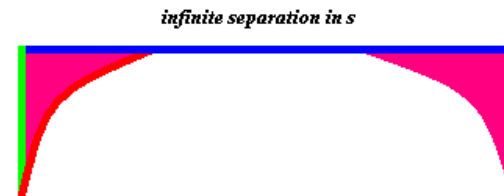
Kaplan, independently also suggested 1)

# Review V

Using this scheme one proceeds to the vector-like case:

- 0) Make wall separation infinite, but now keep both wall vicinities.
- 1) Take  $M_+$  to infinity -  
-can ignore half of the circle.

Fig. 2



# Review VI

This produces the domain wall fermions.

Shamir arrives at 1) directly, without introducing first a finite  $M_+$  followed by taking  $M_+$  to infinity. That these two ways are equivalent follows from replaying the derivation of MIT bag boundary conditions in the 70's.

# Review VII

Comment:

Keeping  $M_+$  finite doubles memory for fermions in standard implementations but the extra numerical cost is minimal in terms of operation counts, since the inversion of the Dirac operator will converge rapidly on the large  $M_+$  side.

However, so far, no numerical advantage has been identified in keeping  $M_+ < \infty$ ; there could be some however, since the wave functions for the zero modes become smoother for finite masses.



# Review VIII

There are 2 options now:

- 1) Work out analytically the infinite wall separation case (NN,N) and implement the result numerically  $\rightarrow$  overlap operator:

$$D_o = \frac{1 + \gamma_5 \text{sign}(H_W)}{2}$$

# Review IX

2) Latticize the 5-th dimension and stick to the 5D Dirac operator which is simple.

In case 1) we aim for chiral symmetry to machine accuracy. In case 2) chiral symmetry is broken and we need to characterize the amount of breaking.

# Review X

This can be done in two ways:

2A) Use EFT logic to assume that the breaking, being small, can be parametrized by  $m_{\text{eff}}$  with

$$L_{eff} = \dots + \bar{\psi} m_{eff} \psi$$

The effective mass can then be evaluated by looking only at one convenient observable.

# Review XI

2B) By numerical analysis logic:

The finite wall separation implies:

$$\textit{sign}(H_w) \rightarrow \textit{smooth}(H_w)$$

violation of chirality is given by

$$||( \textit{smooth}(H_W))^2 - 1 ||$$

# Review XII

Both 2A) and 2B) lead to quantities defined for each gauge background and therefore fluctuating. Moreover, matrix elements of the operator in 2B) have very disparate orders of magnitude, raising doubts about when exactly one can rely on 2A).

## Review XIII

One could replicate the effect by making  $m_{\text{eff}}$  space-time dependent, e.g.

$$m_{eff} \rightarrow m_{eff} + \xi \gamma_5 \text{tr} F(x) \tilde{F}(x)$$

Remember, the EFT logic has to be latticy, since we cannot automatically assume proximity to the continuum limit as a result of the exceptional small modes (BNN).

# Review XIV

There is no doubt that in order to be safe one should use 2B rather than 2A (TW Chiu).

The problem disappears if:

- 1) One goes very close to the continuum limit at finite physical volume.
- 2) One filters out some UV noise from the gauge field.

# Review XV

However:

- 1) Is not yet practical
- 2) Is unsafe, causing large cutoff effects: e.g. one no longer sees a Coulomb force law at small lattice separations.

Actually, the problems are related in the sense that a very large lattice volume would make them both go away.



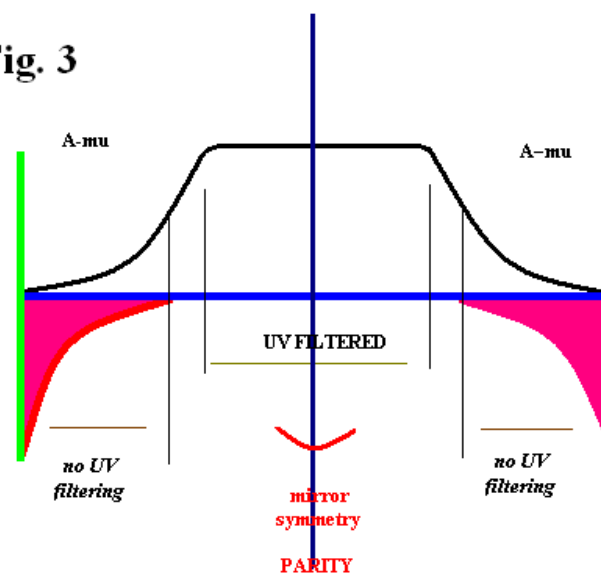
# The main idea

Reinstate the  $s$  dependence of  $A_\mu(x_v, s)$  and possibly of  $A_5(x_v, s)$  but without introducing new fluctuating bosonic gauge fields. Rather, the new five dimensional gauge field is completely determined by one quantum gauge field  $A_\mu(x_v)$ .

# The set-up

- A) Use  $M_+ = \infty$ .
- B) Simplify to the gauge  $A_5 = 0$ .

Fig. 3



# The 5D action

$$D = \begin{pmatrix} C_1^\dagger & B_1 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ B_1 & -C_1 & -1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & -1 & C_2^\dagger & B_2 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & B_2 & -C_2 & -1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & C_3^\dagger & B_3 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & B_3 & -C_3 & \vdots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & B_S & -C_S \end{pmatrix}$$

# The matrices C and B

$$\begin{aligned}
 (C_s)_{x\alpha i, y\beta j} &= \frac{1}{2} \sum_{\mu=1}^d \sigma_{\mu}^{\alpha\beta} [\delta_{y, x+\hat{\mu}} (U_{\mu}^s(x))_{ij} - \delta_{x, y+\hat{\mu}} (U_{\mu}^{s\dagger}(y))_{ij}] \equiv \sum_{\mu=1}^d \sigma_{\mu}^{\alpha\beta} (W_{\mu}^s)_{xi, yj} \\
 (B_{s0})_{x\alpha i, y\beta j} &= \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^d [2\delta_{xy} \delta_{ij} - \delta_{y, x+\hat{\mu}} (U_{\mu}^s(x))_{ij} - \delta_{x, y+\hat{\mu}} (U_{\mu}^{s\dagger}(y))_{ij}] \\
 (B_s)_{x\alpha i, y\beta j} &= (B_{s0})_{x\alpha i, y\beta j} + M_0^s \delta_{x\alpha i, y\beta j}
 \end{aligned}$$

The Gamma matrices are taken in the chiral basis and the  $\sigma_{\mu}$  are their off diagonal 2 by 2 entries.

# The effective 4D action I

$$\det D = (-)^{qk} (\prod_{s=1}^S \det B_s) \det \left[ \frac{1-\Gamma_5}{2} - \mathbf{T}_l \frac{1+\Gamma_5}{2} \right]$$

Following  
the  
standard  
method  
(N) we  
find

$$T_s \equiv e^{-H_s} = \begin{pmatrix} \frac{1}{B_s} & \frac{1}{B_s} C_s \\ C_s^\dagger \frac{1}{B_s} & C_s^\dagger \frac{1}{B_s} C_s + B_s \end{pmatrix}$$

$\mathbf{T}_l$  is a symmetric product of  $T_s^{-1}$  factors. By definition, gauge fields and mass parameters  $M_0$  labeled by  $s$  and  $S - s$  are identical.

$$\mathbf{T}_l = R^\dagger T_r^{-(l+2)} R$$

The  $R, R^\dagger$  factors are complex matrices representing “ramps”:

$$R = T_r^{-1} T_{r-1}^{-1} \dots T_1^{-1}, \quad R^\dagger = T_S^{-1} T_{S-1}^{-1} \dots T_{S-r}^{-1}$$

# The effective 4D action II

Finally we  
get what  
should be  
the new  
overlap  
Dirac  
operator  
with UV  
filtering.

$$\det D = (-)^{qk} (\prod_{s=1}^S \det B_s) \det [1 + \mathbf{T}_1^{-1}] \det \left[ \frac{1 + \Gamma_5 \frac{1 - \mathbf{T}_l}{1 + \mathbf{T}_l}}{2} \right]$$

The same derivation for Pauli Villars (PV) fields would give a PV determinant made out of the first two factors in the formula above. Dividing the two expressions, leaves us with the last factor as representing the almost massless Dirac fermion.

Thus, the operator  $E_l = \frac{1 - \mathbf{T}_l}{1 + \mathbf{T}_l}$  carries the  $l$  dependence and its spectrum governs the approach to the chiral limit  $l = \infty$ .

# On large separation convergence

- Unlike in the homogeneous case, we do not have a sign function of a well defined, sparse Hermitian matrix; nevertheless, from the dwf point of view the simulation will not be more expensive than that of the homogeneous case – this exploits the special features of Kaplan's proposal.
- However, there is a theoretical price to pay:  
Convergence to a finite limit as the separation diverges seems difficult to prove rigorously, and may not hold for EVERY gauge configuration.

# Testing the method

To test, we want to find the chirality violation by the norm characterization. Finding the eigenvalues closest to unity of the combined transfer matrix is tough because the condition number becomes very bad as the number of slices increases. Simple methods work for only very few slices.



# A trick

A trick  
around  
the  
condition  
number  
issue is

It is easy to check that the spectrum of  $B_S =$

$$\begin{pmatrix} 0 & A_1 & 0 & \dots & \dots & 0 \\ 0 & 0 & A_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & A_{S-1} \\ A_S & \vdots & \vdots & \vdots & \vdots & 0 \end{pmatrix}$$

is given by

$$\lambda_{n,k} = \rho_n^{1/S} e^{\frac{i\phi_n}{S}} e^{\frac{2\pi i k}{S}}, \text{ with } k = 0, 1, 2, \dots, S-1$$

where  $\lambda_n = \rho_n e^{i\phi_n}$  are the eigenvalues of  $\Pi_S \equiv A_1 \dots A_S$ .

# In progress

- This is where we are at the moment.
- We hope to return to the problem over the summer, during my planned visit to Berlin.
- There is room for a lot of variation.
- The essence of the idea is to UV filter the gauge fields seen by the heavy modes, which reside mostly in the bulk, while allow the physical fermions at the walls see the unadulterated gauge field. Domain wall UV filtering is a bit like chemotherapy: it destroys more the bad guys.